5.2.3

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Let n be a non-negative integer. Without using calculus, prove that

$$n(1+y)^{n-1} = \sum_{r=1}^{n} \binom{n}{r} r y^{r-1}$$

Hint: Start by substituting $n-1$ for n in the binomial theorem. Then multiply both sides of the equation by n.

Applying the hint, we start from the binomial theorem:

$$(1+y)^n = \sum_{r=0}^{n} \binom{n}{r} 1^r y^{n-r}$$

Substituting $n-1$ for n we get:

$$(1+y)^{n-1} = \sum_{r=0}^{n-1} \binom{n-1}{r} 1^r y^{n-1-r}$$

Multiplying both sides by n we get:

$$n(1+y)^{n-1} = n \sum_{r=0}^{n-1} \binom{n-1}{r} 1^r y^{n-1-r}$$

Let's now pull the n into the sum and change the combination to factorial form to see if there is anything we can merge/do to get the binomial looking like the final $\binom{n}{r}$ we want.

$$n(1+y)^{n-1} = \sum_{r=0}^{n-1} n \left(\frac{(n-1)!}{r!(n-1-r)!} \right) 1^r y^{n-1-r}$$

Clearly the n multiplied by the $(n-1)!$ will give $n!$ in the numerator. But what about the denominator? What can we do to make that the same as it would need to be for $\binom{n}{r}$? We would need to make the right factor from:

$$(n-1-r)(n-2-r)(n-3-r)\cdots(2)(1)$$

Into:

$$(n-r)(n-1-r)(n-2-r)(n-3-r)\cdots(2)(1)$$

So we'd need another $\dfrac{1}{n-r}$ factor multiplied in. To balance, we'd have to multiply by $n-r$ as well. I'm also going to leave off the left side of our equation for now since it already looks right and take out the 1^r factor which is always just 1, anyway:

$$\sum_{r=0}^{n-1} \left(\frac{n-r}{n-r}\right) \left(\frac{n!}{r!(n-1-r)!}\right) y^{n-1-r}$$

Bringing that denominator inside the factorial expression gives us:

$$\sum_{r=0}^{n-1} (n-r) \left(\frac{n!}{r!(n-r)!}\right) y^{n-1-r}$$

Now that is our desired binomial:

$$\sum_{r=0}^{n-1} (n-r) \binom{n}{r} y^{n-1-r}$$

Next let's tackle changing that summation index from a 0-start to a 1-start. Examining the terms in the sequence, we see:

$$(n-0)\binom{n}{0}y^{n-1-0} + (n-1)\binom{n}{1}y^{n-1-1} + (n-2)\binom{n}{2}y^{n-1-2}$$
$$\cdots + (n-(n-2))\binom{n}{n-2}y^{n-1-(n-2)} + (n-(n-1))\binom{n}{n-1}y^{n-1-(n-1)}$$

More simply:

$$n\binom{n}{0}y^{n-1} + (n-1)\binom{n}{1}y^{n-2} + (n-2)\binom{n}{2}y^{n-3}$$
$$\cdots + 2\binom{n}{n-2}y^{1} + 1\binom{n}{n-1}y^{0}$$

Taking a cue from our target equation, we can let r be in $[1..n]$ and let the exponent on y be $r-1$. The first factor becomes r as well, nicely enough! But the bottom of the binomial becomes $n-r$ which seems at first intractable:

$$\sum_{r=1}^{n} r \binom{n}{n-r} y^{r-1}$$

Then, remembering symmetry, we have $\binom{n}{r} = \binom{n}{n-r}$ and we are done:

$$\sum_{r=1}^{n} \binom{n}{r} r y^{r-1}$$

Pulling it all together we see that:

$$n(1+y)^{n-1} = \sum_{r=1}^{n} \binom{n}{r} r y^{r-1}$$

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